§10 INTEGRALITY CONDITION

We now study in detail the question of existence and uniqueness of prequantum line bundles on a symplectic manifold (M,ω).

We come back to the integrality condition of section 6 and its more explicit explanation in section 9A:

[G3] There exists an open cover $(U_j)_{j\in I}$ of M such that the class $[\omega]\in H^2_{dR}(M,\mathbb{C})$ contains (as a Čech class $[\omega]\in H^2((U_j)_{j\in I},\mathbb{C})\cong \check{H}^2(M,\mathbb{C})$) a cocycle $c=(c_{ijk})$, with $c_{ijk}\in\mathbb{Z}$ for all $i,j,k\in I$ with $U_{ijk}\neq\emptyset$.

(10.1) <u>PROPOSITION</u>: Let ω be a given closed two form $\omega\in\Omega^2(M)$ which satisfies [G3]. Then there exists a line bundle with connection ∇ such that $\operatorname{Curv}(L,\nabla)=\omega$.

□ <u>Proof</u>. Without loss of generalization we can assume that all intersections $U_{j_0j_1\dots j_p}=U_{j_0}\cap U_{j_1}\cap\dots\cap U_{j_p}$ are empty or they are contractible (e.g. diffeomorphic to convex open subsets of $\mathbb{R}^n$), so that we can apply the Lemma of Poincaré repeatedly: Since ω is closed there are

$\alpha_j \in \Omega^1(U_j)$ such that $d\alpha_j = \omega$ (Lemma of Poincaré in U_j).
On $U_{jk} = U_j \cap U_k \neq \emptyset$ the one forms $\alpha_j - \alpha_k$ are closed:

$$d\alpha_j - d\alpha_k = \omega - \omega = 0.$$

Hence, there exist $f_{jk} \in \mathcal{E}(U_{jk})$ with $df_{jk} = \alpha_j - \alpha_k$ (Lemma of Poincaré). Because of

$$d\,(f_{jk} + f_{ki} + f_{ij}) = 0$$

we finally obtain constants $c_{ijk} \in \mathbb{R}$ (not that U_{jk} is empty or connected) with

$$c_{ijk} = f_{ij} + f_{jk} + f_{ki} \quad \text{on } U_{ijk} \neq \emptyset.$$

Now, the condition [G3] implies that there are (z_{ijk}) with $z_{ijk} \in \mathbb{Z}$ and and $x_{jk} \in \mathbb{R}$ such that

$$z_{ijk} = c_{ijk} + x_{ij} + x_{jk} + x_{ki}, \quad \text{if } U_{ijk} \neq \emptyset.$$

In the case of $U_{jk} \neq 0$ we set

$$g_{jk} := \exp\,(2\pi i f_{jk} + 2\pi i x_{jk}) \in \mathcal{E}(U_{jk}, \mathbb{C}^\times).$$

We immediately conclude

$$g_{ij}\, g_{jk}\, g_{ki} = \exp\,(2\pi i\,(c_{ijk} + x_{ij} + x_{jk} + x_{ki})) = 1$$

on U_{ijk}, hence the smooth functions

$$g_{jk} : U_{jk} \to \mathbb{C}^\times, \quad j, k \in I \text{ with } U_{jk} \neq \emptyset$$

defines a complex line bundle L over M according to proposition (3.7). The forms α_j define a connection over each U_j with

$$\alpha_j - \alpha_k = df_{jk} = \frac{1}{2\pi i}\,\frac{dg_{jk}}{g_{jk}}\,.$$

Therefore, the α_j are local gauge potentials of a connection ∇ (cf. (4.3)) with ω as its curvature since $\omega|_{U_j} = d\alpha_j$ (cf. (6.2)). $\square$

The line bundle with connection which we have just constructed has a compatible Hermitian structure H. (the α_j are real forms, see (7.5) for the compatibility). As a result, we obtain a prequantum line bundle (L, ∇, H).

More precisely, we have the chance to understand (10.1) and the proof of (10.1) for real forms $\omega \in \Omega^2(M)$ and more generally, for complex forms. In the real case (which we have assumed in (10.1) by the notation $\omega \in \Omega^2(M) = \Omega^2(M,\mathbb{R})$) it is clear that the α_j's can be chosen real valued as well and hence allow a compatible Hermitean structure. The scheme of proof works, however, also for the complex forms $\omega \in \Omega^2(M,\mathbb{C}) = \Omega^2(M)\otimes\mathbb{C}$.

(10.2) REMARK. As we mentioned in section 6 the integrality conditions [G1] and [G3] are equivalent (A. Weil's theorem). We have given a complete proof of the implication

„[G3] $\Rightarrow$ [G1]":

In fact by (10.1) there exists a line bundle L over M with connection ∇ such that $\mathrm{Curv}(L,\nabla) = \omega$, - if ω satisfies [G3]. And by (6.5) ω satisfies [G1], i.e. $\int_S \omega \in \mathbb{Z}$ for all oriented compact closed surfaces.

The converse "[G1] $\Rightarrow$ [G3]" can be carried through along the same lines of ideas as the proof of (10.1).

Uniqueness:

After the question of existence we now discuss the uniqueness of line bundles with connection with given curvature form $\omega \in \Omega^2(M)$.

In other words:
What freedom do we have in constructing (L,∇,H)? How many different or inequivalent prequantum line bundle exist on (M,ω)? Under which

conditions is the prequantum bundle essentially unique?

First, we can replace α_i by α_i' in the above construction, i.e. $\alpha_i' \in \Omega^1(U_i)$ with $d\alpha_i' = \omega|_{U_i}$. Therefore,

$$\alpha_j' = \alpha_j + dl_j \quad , \; l_j \in \mathcal{E}(U_j),$$

so that f_{ij} may be replaced by

$$f_{ij}' = f_{ij} + l_i - l_j :$$

$$df_{ij}' = d(f_{ij} + l_i - l_j) = \alpha_i - \alpha_j + dl_i - dl_j = \alpha_i' - \alpha_j' .$$

In the next step we can use the same c_{ijk} as before:

$$c_{ijk} = f_{ij} + f_{jk} + f_{ki} \qquad (\in \mathbb{R} \text{ constant see above})$$

$$= (f_{ij} + l_i - l_j) + (f_{jk} + l_j - l_k) + (f_{ki} + l_k - l_i).$$

$$= f_{ij}' + f_{jk}' + f_{ki}' .$$

Hence the corresponding transition functions g_{jk}' are

$$g_{jk}' := \exp\left(2\pi i\,(f_{jk} + l_j - l_k) + 2\pi i\, x_{jk}\right)$$

with $g_{jk}' g_{ki}' g_{ij}' = 1$ and

$$g_{jk}' = g_{jk} \frac{g_j}{g_k} \quad , \text{ where } g_j = \exp(2\pi i\, l_j) .$$

As a consequence, the line bundle L' defined by the cocycle (g'_{jk}) is equivalent to the line bundle L defined by (g_{jk}): The 2 cocycles are equivalent in $H^1(\mathfrak{U}, \mathcal{E}^*)$. Moreover, if ∇ is the connection on L given by (α_j) and ∇' is the connection on L' given by (α'_j) then ∇' is carried over to ∇ by the equivalence of L' and L (induced by the g_j).

Secondly, fixing α_j with $d\alpha_j = \omega|_{U_j}$, we can replace each f_{jk} by $f_{jk} + b_{jk}$ with real constants $b_{jk} \in \mathbb{R}$. Then the b_{jk} are absorbed into the x_{jk}'s so again we stay in the same equivalence class.

Finally, we can replace (now fixing α_j and f_{jk}) the constants x_{jk} by $x_{jk} + 2\pi y_{jk}$, where the y_{jk} are constants $y_{jk} \in \mathbb{R}$ with

$$y_{ij} + y_{jk} + y_{ki} = \mathbb{Z} \quad \left(\text{all } i,j,k \in I, \text{ with } U_{ijk} \neq \emptyset\right)$$

and $y_{jj} = 0$, as well as $y_{jk} + y_{kj} = 0$. If we replace g_{jk} by

$$g''_{jk} = g_{jk} \exp(2\pi i\, y_{jk}) = g_{jk}\, t_{jk}, \quad t_{jk} = e^{2\pi i y_{jk}}.$$

(g''_{jk}) defines a line bundle L''. The collection (t_{jk}) induces a cohomology class

$$[(t_{jk})] \in H^1(\mathfrak{U}, \mathbb{S}) = \check{H}^1(M, \mathbb{S}),$$

where $\mathbb{S} = \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the circle group. If (and only if) the class $[(t_{jk})]$ is trivial in $H^1(\mathfrak{U}, \mathbb{S})$ the corresponding line bundle L'' is equivalent to L taking ∇'' (given by v_j) to ∇: $[(t_{jk})]$ being trivial means $t_{jk} = \lambda_j \lambda_k^{-1}$ for $\lambda_j \in \mathbb{S}$ constant and therefore

$$g''_{jk} = g_{jk} \lambda_j \lambda_k^{-1}.$$

Conversely, if L'' and L are equivalent as line bundles with connection there are $\lambda_j \in \mathbb{S}$ with $t_{jk} = \lambda_j \lambda_k^{-1}$, i.e. $[(t_{jk})] = 1 \in \check{H}^1(M, \mathbb{S})$. We have shown:

(10.3) <u>PROPOSITION</u>: Let $\omega \in \Omega^2(M)$ satisfy the integrality condition [GS]. Then the set of equivalence classes of line bundles L with connection ∇ with $\mathrm{Curv}(L,\nabla) = \omega$ is in one to one correspondence with $H^1(M, \mathbb{S})$.

In particular,

(10.4) <u>PROPOSITION</u>: Let M be simply connected and let $\omega \in \Omega^2(M)$ satisfy [GS]. Then there exists exactly one line bundle L with connection such that $\mathrm{Curv}(L, \nabla) = \omega$ up to equivalence.

$\pi_1(M) = 0$ implies $H^1(M, \mathbb{S})$ [*]

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[*] Übung. What about $H^1(M, G)$, G any abel. group?

(10.5) COROLLARY: A simply connected quantizable symplectic manifold (M, ω) has exactly one prequantum bundle (up to equivalence).

REMARK: It is possible that L and L'' are equivalent as line bundles but not as line bundles with connection. [*]

(10.6) REMARK: Let M be a manifold and let $\mathcal{U} = (U_j)_{j \in I}$ be an open cover such that all $U_{j_0 \dots j_n}$ are contractable. The transition functions (g_{jk}) of any complex line bundle L over M with respect to the cover $\mathcal{U}$ (always exist) and they define a cohomology class in

$$H^1(\mathcal{U}, \mathbb{Z}) = \check{H}^1(M, \mathbb{Z}),$$

hence an entire cohomology class.
In fact, let

$$z_{ijk} := \frac{1}{2\pi i}\left(\log g_{ij} + \log g_{jk} + \log_{ki}\right) \text{ on } U_{ijk}.$$

Locally z_{ijk} is well-defined and integer-valued (since $e^{2\pi i z_{ijk}} = g_{ij} g_{jk} g_{ki} = 1$). Therefore, (z_{ijk}) defines an element in $C^2(\mathcal{U}, \mathbb{Z})$. There is a problem in getting a global interpretation of the z_{ijk}, due to the fact that the logarithm is ambigouos in $\mathbb{C}^\times$;

[*] Übung: Give examples

but the corresponding Čech cohomology class

$$\left[(z_{ijk}) \right]$$

in $\check{H}^2(\mathfrak{U}, \mathbb{Z}) = \check{H}^2(M, \mathbb{Z})$ is well-defined.

The class $c(L) := \left[(z_{ijk}) \right] \in \check{H}^2(M, \mathbb{Z})$ is called the CHERN CLASS of the line bundle L. $c(L)$ is an important invariant of the (equivalence class) of the line bundle!

As an immediate consequence we obtain:

(10.7) <u>PROPOSITION</u>: A symplectic manifold is quantizable if and only if the symplectic form satisfies the integrality condition [G3].

We know this result already by using Weil's theorem

"[G1] <=> [G3]",

but (10.7) can also be deduced from the above results in the following way:

- "[G3] $\Rightarrow \exists$ prequantum bundle" is (10.1)
- the converse follows from the last remark.